Understanding GW150914

Pierre Binétruy, Université Paris Diderot,
http://gravity.paris
February 11, 2015

It is rather easy to understand some of the characteristics of the gravitational event observed by the LIGO detector on September 14, 2015, with only knowledge of classical mechanics and a little input from general relativity. If you are an undergraduate student in physics with an understanding of basic classical mechanics, this is intended for you.

Let us start by recalling Kepler's law.

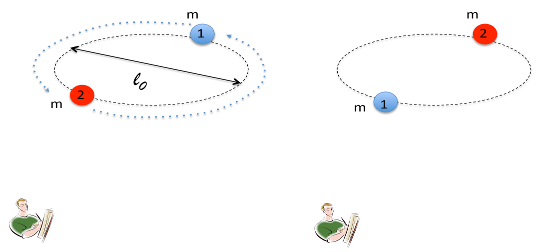

Figure 1: Binary system of two astrophysical objects of mass m

We consider a binary system of two astrophysical objects of identical mass m in circular orbit around one another at a distance l_0 (see Figure 1). The angular velocity ω_s (subscript s for source) is obtained from the standard equation which relates the gravitational force on one of the objects to the centrifugal acceleration:

$$\frac{G_N m^2}{l_0^2} = m\omega_s^2 \frac{l_0}{2} \ , \tag{1}$$

where G_N is Newton's constant. This yields Kepler's law:

$$\omega_s^2 l_0^3 = G_N 2m \quad \text{or} \quad \omega_s = \sqrt{\frac{G_N 2m}{l_0^3}} \ . \tag{2}$$

Note from Figure 1 that, from the gravitational point of view, an observer sees the same gravitational system when the objects have rotated by half a turn. Hence (and this can be proven explicitly), the frequency of the gravitational waves emitted $\omega = 2\pi f$ is just twice the frequency of rotation:

$$\omega = 2\omega_s \quad \text{or} \quad f = \frac{\omega_s}{\pi} \ . \tag{3}$$

Hence, and this is important to remember in what follows, detecting a gravitational wave at frequency f gives us information on the rotational velocity of the binary system at the time the gravitational wave was emitted.

In the case of two objects of unequal masses m_1 and m_2, one may note that, classically, the Lagrangian of the system $L = T - V$ may simply be written in terms of the motion of the centre of mass of the binary system ($\mathbf{R} \equiv \mathbf{r_1} + \mathbf{r_2}$) and of the relative motion ($\mathbf{r} \equiv \mathbf{r_1} - \mathbf{r_2}$):

$$L = \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} \mu \dot{\mathbf{r}}^2 + G_N \frac{\mu M}{r} \ , \tag{4}$$

where a dot means a time derivative and I have defined the total mass M and the reduced mass μ:

$$M \equiv m_1 + m_2 \ , \quad \mu \equiv \frac{m_1 m_2}{m_1 + m_2} \ . \tag{5}$$

The overall motion does not generate gravitational waves. The reason is that in order to generate curvature waves, you need to consider two distant points[1]. Hence the motion of the centre of mass does not generate gravitational waves. On the oher hand, regarding the relative motion, we see by defining $\hat{r} = \mu^{1/2} r$, that it depends on the masses only through the mass parameter

$$\mathcal{M} = \left(\mu^3 M^2\right)^{1/5} = \left(\frac{m_1^3 m_2^3}{m_1 + m_2}\right)^{1/5} \ . \tag{6}$$

[1]There is no meaning to curvature if you consider a single point in space-time. In other words, any curved space-time is localy flat. This is the essence of what is known as Einstein's *equivalence principle*: if I stay at a given point in space-time, I can construct locally coordinates where everything seems flat.

This is the only dimensionful parameter characterising the binary system from the point of view of gravitational waves. It is called the *chirp mass* for reasons to be explained just below. In the case of GW150914, since $m_1 = 26M_{\odot}c^2$ and $m_1 = 36M_{\odot}c^2$, one obtains $\mathcal{M} = 28M_{\odot}c^2$.

Now, we have to implement the fact that the system is emitting gravitational waves and is thus losing energy. Thus the distance between the objects decreases from the initial value l_0 (we will thus call l the distance at any given time), the rotational velocity increases because of Kepler's law (2). Hence the frequency of the emitted wave increases: one calls this phenomenon *chirping*, in reference to the sound waves emitted by a bird who chirps.

But how much energy is lost by the system per unit of time? How much power is lost in the form of gravitational waves? Well, Einstein computed this for us in his 1918 paper. This is the famous quadrupole formula. We will not prove it here but accept it as an input from General Relativity (see for example the book of Michele Maggiore, Gravitational Waves, volume 1, Oxford University Press). Let me give this formula in a slightly unconventional way. The power *i.e.* energy per unit of time, carried away by gravitational waves id given by

$$P_{_{GW}} \sim \frac{G_{_N}}{c^5} P_{\rm int}^2 \ . \tag{7}$$

In here, $P_{\rm int}$ is the gravitational power flow internal to the source: for a system of size R which involves a mass M moving in a time T, we have $P_{\rm int} \sim M(R/T)^2/T$, Let me note that $c^5/G_{_N}$ has obviously the dimensions of a power as well. It gives the order of magnitude of the power released by such events: $c^5/G_{_N} \sim 3.6 \times 10^{52}$ W $\sim 2 \times 10^5\ M_{\odot}c^2/\text{s}$!

Let us return now to our example of two astrophysical objects of mass m_1 and m_2 orbiting around one another at a distance l, with angular velocity ω_s. The total energy is the sum of the kinetic energy and of the gravitational potential energy $E = E_k + E_p$. Using the virial theorem $E_k = -E_p/2 = G_{_N}\mu M/(2l)$ and Kepler's law (2) with $2m$ replaced by M (and l_0 by l), one finds

$$E = -E_k = E_p/2 = -\frac{G_{_N}}{2}\frac{\mu M}{l} = -\frac{1}{2}G_{_N}^{2/3}\omega_s^{2/3}\mathcal{M}^{5/3} \ , \tag{8}$$

By dimensional analysis, the internal power flow $P_{\rm int}$ is a typical energy, say the kinetic energy E_k, times an inverse of time, say the frequency ω_s. Hence, from the quadrupole formula (7), one easily obtains

$$P_{_{GW}} = \frac{32}{5c^5} G_{_N}^{7/3} \left(\omega_s \mathcal{M}\right)^{10/3} \ , \tag{9}$$

where I have restored the right numerical coefficient in front.

But this is the energy lost by the binary system per unit time: $dE/dt = -P_{_{GW}}$. We thus obtain from the last two equations

$$\frac{dE}{dt} = -\frac{32}{5c^5} G_{_N}^{7/3} \left(\omega_s \mathcal{M}\right)^{10/3} = -\frac{1}{3} G_{_N}^{2/3} \mathcal{M}^{5/3} \omega_s^{-1/3} \frac{d\omega_s}{dt} \ . \tag{10}$$

This equation can be solved for ω_s. We write this solution in terms of the gravitational wave frequency $f = \omega_s/\pi$:

$$f = \frac{1}{8\pi} \left[\frac{5c^5}{G_{_N}^{5/3} \mathcal{M}^{5/3}} \right]^{3/8} \frac{1}{(t_m - t)^{3/8}} \ . \tag{11}$$

where t_m is the time where the two objects merge; hence $\tau \equiv t_m - t$ is the time before merger, before the final touchdown. We see that the frequency increases with time (it chirps) and that the key parameter is the chirping mass $\mathcal{M}$.

We can write this equation in the following form, which is very useful for discussing the GW150914 signal:

$$f \sim 18 \text{ Hz} \left(\frac{30 M_\odot}{\mathcal{M}} \right)^{5/8} \left(\frac{1\text{s}}{\tau} \right)^{3/8} \tag{12}$$

or

$$\tau \sim 4.8 \text{ s} \left(\frac{30 M_\odot}{\mathcal{M}} \right)^{5/3} \left(\frac{10\text{Hz}}{f} \right)^{8/3} \ . \tag{13}$$

This means that the detector picks up the signal of the order of one second before the merger, at 100 Hz, one is 10 millisecond before the merger. Of course, this only gives an order of magnitude and more refined calculations are in order because, when the two black holes become closer, the quadrupole formula is no longer valid and higher multipoles as well as relativistic corrections are necessary (the velocities are no longer negligible with respect to the velocity of light c). But this allows to get a general understanding of the orders of magnitude, as you can verify by looking at the GW160914 discovery paper.

In the case of the space detector eLISA, which will operate in the frequency window $[10^{-4}\text{Hz}, 10^{-1}\text{Hz}]$ and will observe supermassive black holes of a few million solar masses, we can write (11) instead in the following form:

$$\tau \sim 2 \times 10^6 \text{ s} \left(\frac{10^6 M_\odot}{\mathcal{M}}\right)^{5/3} \left(\frac{10^{-4}\text{Hz}}{f}\right)^{8/3} \tag{14}$$

This means that one picks up the signal a few months before the merger, which allows time to predict the date and time of merger, as well as an approximate location in the sky.

References

Gravitationswellen,
A. Einstein, Sitzungsberichte, Preussische Akademie der Wissenschaften, 154 (1918)

Observation of gravitational waves from a binary black hole merger,
B.P. Abbott *et al.*, LIGO Scientific Collaboration and Virgo Collaboration, Phys.Rev. Lett. 116, 061102 (2016)